Quantum limits to information about states for finite dimensional Hilbert space

1991 J. Phys. A: Math. Gen. 24 121

(http://iopscience.iop.org/0305-4470/24/1/021)

J. Phys. A: Math. Gen. 24 (1991) 121–130. Printed in the UK

Quantum limits to information about states for finite dimensional Hilbert space

K R W Jones†

H H Wills Physics Laboratory, University of Bristol, Royal Fort, Tyndall Avenue, Bristol BS8 1TL, UK

Received 18 May 1990

Abstract. A refined bound for the correlation information of an N-trial apparatus is developed via an heuristic argument for Hilbert spaces of arbitrary finite dimensionality. Conditional upon the proof of an easily motivated inequality we are then able to find the optimal apparatus for large ensemble quantum inference, thereby solving the asymptotic optimal state determination problem. In this way we are able to identify an alternative inferential uncertainty principle, which is then contrasted with the usual Heisenberg uncertainty principle.

1. Introduction

Quantum inference [1] is a formalism that employs elements of communication theory [2] to constrain information available about the quantum state of an ensemble containing many identically prepared systems. For a detailed discussion of this idea see [3], its principal precursor is [4].

In [3] we constructed natural inferred distributions upon finite dimensional Hilbert space as the results of state determination *gedanken* experiments. These have confidence limits imposed by the geometry of Hilbert space and the number, N, of examinable ensemble members.

The existence of such limits is easy to establish [3], drawing them precisely leads to a difficult optimization problem upon the geometry of Hilbert space. This is the optimal state determination problem [3] (hereafter OSDP). Here it is solved in the asymptotic regime of large N, conditional upon the proof of an easily motivated conjecture.

2. Review of the OSDP

The aim is to constrain knowledge of the ensemble state through establishing limits to confidence in the measurement of states. To do so we focus attention upon an ensemble of identical pure states. For simplicity a restriction is made to finite dimensional Hilbert space, so the unknown state is labelled $\psi \in C^d$, with d the dimension.

† Present address: School of Physics, University of Melbourne, Parkville 3052, Melbourne, Australia.

This provides circumstances of maximal possible initial information. To obtain maximal information extraction, we consider the highly idealized possibility of making individual observations of each ensemble member with its own Hermitian operator. At this level the result of observation of the kth ensemble member is some single eigenvector, ϕ_k, belonging to the kth measurement operator. These are recognized by their associated eigenvalues, which need not appear explicitly.

In this way each choice of N Hermitian operators $\hat{A}_k$ defines a generalized N-trial state determination apparatus, denoted $\mathscr{A}_N$. Only the eigenvectors of the $\hat{A}_k$ are important, so possibly many choices of N operators map to one $\mathscr{A}_N$. Dropping all reference to the operators, $\mathscr{A}_N$ is simply defined to be a collection of N orthonormal bases for C^d.

Data from a single experiment with $\mathscr{A}_N$ is a set of N eigenvector outcomes, $\Phi_N \equiv \{\phi_k\}_{k=1}^N$. There are d^N possible outcomes and these occur with conditional probability:

$$p(\Phi_N|\psi) \equiv \prod_{k=1}^{N} |\langle\psi|\phi_k\rangle|^2. \tag{1}$$

This represents a correlation between the reading Φ_N of $\mathscr{A}_N$ and the unknown state ψ. This correlation can be inverted via Bayes' rule to obtain $p(\psi|\Phi_N)$.

A representation independent inversion is obtained by specifying a unitary invariant prior distribution on state space [3]. This we call the quantum invariant prior and it is realized as the normalized ray measure

$$\mathrm{d}\hat{\Omega}_{\bar{\psi}} \equiv (d-1)!\,\delta(1-\bar{\psi}\psi)\,\mathrm{d}\bar{\psi}\,\mathrm{d}\psi \tag{2}$$

where $\mathrm{d}\bar{\psi}\,\mathrm{d}\psi \equiv \Pi_{j=1}^{d}\,\mathrm{d}x_j\,\mathrm{d}y_j/\pi$ and $\bar{\psi}\psi = \langle\psi|\psi\rangle$.

Then quantum inference is the simple set of rules

$$p(\psi|\Phi_N) = \frac{p(\Phi_N|\psi)}{p(\Phi_N)} \tag{3}$$

$$p(\Phi_N) = \int p(\Phi_N|\psi)\,\mathrm{d}\hat{\Omega}_{\bar{\psi}}. \tag{4}$$

Details of the calculation of $p(\Phi_N)$ have been given in [3].

The introduction of concepts drawn from communication theory [2, 5] then enables the performance of any $\mathscr{A}_N$ to be quantified as

$$\{\psi, \Phi_N\}[\mathscr{A}_N] \equiv \sum_{\Phi_N} p(\Phi_N) \int p(\psi|\Phi_N) \log p(\psi|\Phi_N)\,\mathrm{d}\hat{\Omega}_{\bar{\psi}}. \tag{5}$$

This is called the correlation information of the apparatus [3]. It measures the average information in nats that the outcomes Φ_N of $\mathscr{A}_N$ yield about the unknown state ψ.

In [3] it was shown that

$$\{\psi, \Phi_N\}[\mathscr{A}_N] \leq N\left(\log d - \sum_{k=2}^{d} 1/k\right) \tag{6}$$

with d the Hilbert space dimension. This bound establishes the existence of the OSDP, where this is defined as the extremization of $\{\psi, \Phi_N\}$ with respect to the choice of $\mathscr{A}_N$. One such problem exists for each pair (N, d).

This paper will concentrate upon refining (6) through examination of the formula (5). The new bound holds everywhere and allows a particularly nice solution to the

OSDP in the asymptotic regime of large N. The new bound we interpret as a new kind of inferential uncertainty principle.

3. Formula for certain ray space integrals

To speed calculation we first develop further a tool given in [3]. There it was shown that for integrable *seed functions* f of a single real variable, there is the simple ray space integration formula

$$\int f(|\langle\psi|\phi\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\bar{\phi}}=(d-1)!\left\{[f]^{d-1}(1)-\sum_{j=1}^{d-1}\frac{[f]^j(0)}{(d-j-1)!}\right\} \tag{7}$$

where $[f]^m(\lambda)$ indicates the mth iterated antiderivative of f evaluated at the point λ. The linear bracket notation is defined inductively by:

$$[f]\equiv\int^u f(w)\,\mathrm{d}w$$

with $[f]^{m+1}=[[f]^m]$ and $\mathrm{d}/\mathrm{d}u[f]^m=[f]^{m-1}$. Apart from some notational differences, this is essentially Jeffreys' device of treating integration as a linear operator, the Q of his work [6].

Some useful iterated antiderivatives follow:

$$[u^l]^m=\frac{l!}{(l+m)!}u^{l+m}\qquad l\geqslant 0 \tag{8}$$

$$[\log u]^m=\frac{u^m}{m!}\left(\log u-\sum_{k=1}^{m}\frac{1}{k}\right) \tag{9}$$

$$[u^l\log u]^m=\frac{l!\,u^{l+m}}{(l+m)!}\left(\log u-\sum_{k=1}^{m}\frac{1}{l+k}\right)\qquad l\geqslant 1 \tag{10}$$

$$[\mathrm{e}^{Nu}]^m=\frac{1}{N^m}\mathrm{e}^{Nu}. \tag{11}$$

These are readily proved by induction on m using

$$[u^l]=\frac{u^{l+1}}{l+1}\qquad\text{and}\qquad[u^l\log u]=\frac{u^{l+1}}{l+1}\left(\log u-\frac{1}{l+1}\right)$$

where

$$[uf]^m=u[f]^m-m[f]^{m+1}$$

provides a short cut to generating such formulae. It can be proved by induction starting with the rule for integration by parts, $[uf]=u[f]-[f]^2$.

All integrations to be done here shall involve reduction to one of the above cases followed by application of (7) to read off the result. For example it is shown in appendix 1 that:

$$\int|\langle\phi|\omega\rangle|^2 f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\bar{\omega}}=C_\perp\int g_\perp(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\bar{\omega}}+C_\parallel\int g_\parallel(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\bar{\omega}} \tag{12}$$

where:

$$g_\perp(u) \equiv (1-u)f(u) \qquad \text{and} \qquad C_\perp = \frac{1}{d-1}(1-|\langle\phi|\psi\rangle|^2) \tag{13}$$

$$g_\parallel(u) \equiv uf(u) \qquad \text{and} \qquad C_\parallel = |\langle\phi|\psi\rangle|^2. \tag{14}$$

Armed with these formulae we can proceed to the calculation.

4. Refined upper bound

It follows directly from (5) that

$$\{\psi, \Phi_N\}[\mathscr{A}_N] \leq \sup_{\Phi_N} \int p(\psi|\Phi_N) \log p(\psi|\Phi_N)\, \mathrm{d}\hat{\Omega}_{\tilde{\psi}}. \tag{15}$$

Furthermore, since all N-trial distributions assume the generic form

$$p(\psi|\Phi_N) = \frac{1}{\mathcal{N}} \prod_{k=1}^{N} |\langle\psi|\phi_k\rangle|^2$$

it follows that (15) can be replaced by a supremum over the choice of N *variable* eigenvectors,

$$\{\psi, \Phi_N\}[\mathscr{A}_N] \leq \sup_{\{\phi_k\}} \int p(\psi|\Phi_N) \log p(\psi|\Phi_N)\, \mathrm{d}\hat{\Omega}_{\tilde{\psi}}. \tag{16}$$

Consider varying the ϕ_k so as to maximize the right-hand side. Intuition suggests the result $\phi_k = \phi$ for all k, since this generates the most peaked $p(\psi|\Phi_N)$ and it is this property that the information measures.

So it is conjectured that the inferred distribution of maximal information is:

$$p(\psi|\Phi_N) = \frac{1}{\mathcal{N}} (|\langle\psi|\phi\rangle|^2)^N \tag{17}$$

where the choice of ϕ does not matter. We have, as yet, no proof of this claim and proceed upon the basis that there is little doubt of its validity.

To calculate $\mathcal{N}$ use (7) and (8) with $f(u) = u^N$ to obtain:

$$\mathcal{N} = \frac{(d-1)!\,N!}{(N+d-1)!}. \tag{18}$$

Then substitution of (17) into (16) yields:

$$\{\psi, \Phi_N\}[\mathscr{A}_N] \leq \int \frac{1}{\mathcal{N}} (|\langle\psi|\phi\rangle|^2)^N \log\left[\frac{1}{\mathcal{N}} (|\langle\psi|\phi\rangle|^2)^N\right] \mathrm{d}\hat{\Omega}_{\tilde{\psi}} \tag{19}$$

which readily reduces to

$$\{\psi, \Phi_N\}[\mathscr{A}_N] \leq \log\frac{1}{\mathcal{N}} + \frac{N}{\mathcal{N}} \int f(|\langle\psi|\phi\rangle|^2)\, \mathrm{d}\hat{\Omega}_{\tilde{\psi}} \tag{20}$$

where now $f(u) = u^N \log u$. Applying (7) with (10),

$$\int f(|\langle\psi|\phi\rangle|^2)\, \mathrm{d}\hat{\Omega}_{\tilde{\psi}} = -\frac{(d-1)!\,N!}{(N+d-1)!} \left(\frac{1}{N+1} + \ldots + \frac{1}{N+d-1}\right). \tag{21}$$

Substitution of this and (18) into (20) then yields the final result:

$$\{\psi, \Phi_N\}[\mathscr{A}_N] \leqslant \log\left[\frac{(N+d-1)!}{(d-1)!N!}\right] - N\sum_{k=1}^{d-1}\frac{1}{N+k} \tag{22}$$

which provides the sought after bound. It constrains apparatus performance for all N and d.

5. Bounded asymptotics

The bound (22) is exact, for later use we now derive its large N behaviour. This provides a constraint upon the asymptotic form of the correlation information for any apparatus. First observe that

$$\frac{(d+N-1)!}{(d-1)!N!} = \frac{N^{d-1}}{(d-1)!}\prod_{k=1}^{d-1}\left(1+\frac{k}{N}\right) \tag{23}$$

$$N\sum_{k=1}^{d-1}\frac{1}{N+k} = \sum_{k=1}^{d-1}\frac{1}{1+k/N}. \tag{24}$$

Then it is clear that for large N:

$$\log\left\{\frac{(N+d-1)!}{(d-1)!N!}\right\} \to \log\left\{\frac{N^{d-1}}{(d-1)!}\right\} \tag{25}$$

$$-N\sum_{k=1}^{d-1}\frac{1}{N+k} \to -(d-1). \tag{26}$$

In both cases the approach is from above. Adding both terms as in (22) then gives

$$\{\psi, \Phi_N\}[\mathscr{A}_N]_{\text{asymp}} \leqslant \log\left\{\frac{N^{d-1}}{(d-1)!}\right\} - (d-1) \tag{27}$$

where the terms that were discarded vanish in the limit $N \to \infty$, and so merely mediate passage to this asymptotic form for the bound. However, because the approach is from above, this bound is properly one upon the possible asymptotics of the correlation information, not the correlation information itself.

Note that the leading $\log N^{d-1}$ behaviour is generic for Gaussian location upon a $2(d-1)$ dimensional space. This is precisely the dimensionality of the pure state manifold of a d-state system, so the result is not unexpected.

6. The isotropic apparatus

To solve the OSDP in the asymptotic regime it is sufficient to exhibit an apparatus whose asymptotics achieve equality in (27). This will be the optimal apparatus for large N quantum inference.

To characterize that apparatus which achieves the upper bound consider repeated use of m bases n times with $N = nm$. Both n and m are to be taken large. With m large, we can consider the full set of measurement bases to be spread isotropically in Hilbert space, to yield a uniform density of possible eigenvector results. The limiting case as $N \to \infty$ we call the isotropic apparatus.

It is shown in appendix 2 that, with the above limiting procedure implicit, the effective correlation of this idealized apparatus is given by the expression:

$$p(\phi|\psi) \equiv \frac{1}{\mathcal{N}} \exp\left\{ Nd \int |\langle\phi|\omega\rangle|^2 \log|\langle\psi|\omega\rangle|^2 \, \mathrm{d}\hat{\Omega}_{\bar{\omega}} \right\}. \tag{28}$$

Here ϕ is a new outcome parameter being the maximum of $p(\Phi_N|\psi)$ for those outcomes Φ_N which dominate in the sense of large probability of occurrence.

Implicit in the introduction of ϕ is a map $\Phi_N \mapsto \phi$. However, it is not necessary to study this because the inbuilt symmetry of the isotropic apparatus ensures that the maxima ϕ of all dominant inferred distributions will be uniformly distributed.

It follows that for the correlation information the isotropic apparatus we should take:

$$\{\psi, \phi\}[\mathscr{A}^{\mathrm{op}}] \equiv \iint p(\psi|\phi) \log p(\psi|\phi) \, \mathrm{d}\hat{\Omega}_{\bar{\phi}} \, \mathrm{d}\hat{\Omega}_{\bar{\psi}} \tag{29}$$

To calculate this quantity first consider the exponent in (28), namely

$$E \equiv Nd \int |\langle\phi|\omega\rangle|^2 \log|\langle\psi|\omega\rangle|^2 \, \mathrm{d}\hat{\Omega}_{\bar{\omega}}. \tag{30}$$

Notice that this can be handled by the decomposition formula (12) with

$$g_\perp(u) = (1-u)\log u \tag{31}$$

$$g_\parallel(u) = u \log u. \tag{32}$$

Calculating the two component integrals of (12) with (7) using the results (9) and (10), we find, after some manipulation that,

$$E = N|\langle\psi|\phi\rangle|^2 - N \sum_{k=1}^{d} \frac{1}{k}.$$

Returning now to (28), notice that the constant term can be absorbed into the normalization and so

$$p(\phi|\psi) = \frac{1}{\mathcal{N}} \exp\{N|\langle\phi|\psi\rangle|^2\} \tag{33}$$

where

$$\begin{aligned} \mathcal{N} &\equiv \int p(\phi|\psi) \, \mathrm{d}\hat{\Omega}_{\bar{\phi}} \\ &= \int \exp\{N|\langle\phi|\psi\rangle|^2\} \, \mathrm{d}\hat{\Omega}_{\bar{\phi}} \\ &= \frac{(d-1)!}{N^{d-1}} \left\{ \mathrm{e}^N - \sum_{k=0}^{d-2} \frac{N^k}{k!} \right\} \\ &= \frac{(d-1)!}{N^{d-1}} \mathrm{e}^N \left\{ 1 - \mathrm{e}^{-N} \sum_{k=0}^{d-2} \frac{N^k}{k!} \right\} \end{aligned} \tag{34}$$

where use has been made of (7) with $f(u) = \mathrm{e}^{Nu}$ and the result (11).

Note that the symmetric appearance of ϕ and ψ in (33) ensures that:

$$p(\psi|\phi) = p(\phi|\psi).$$

Looking at (33) it becomes obvious that perfect knowledge of the state ψ obtains as $N \to \infty$. Also, it is now straightforward to verify that

$$\{\psi, \phi\}[\mathscr{A}^{\text{op}}] = \iint \frac{1}{\mathscr{N}} \exp\{N|\langle\phi|\psi\rangle|^2\} \log \frac{1}{\mathscr{N}} \exp\{N|\langle\phi|\psi\rangle|^2\} \, d\hat{\Omega}_{\hat{\phi}} \, d\hat{\Omega}_{\hat{\psi}}$$

$$= \log \frac{1}{\mathscr{N}} + \frac{N}{\mathscr{N}} \frac{d}{dN}(\mathscr{N}). \tag{35}$$

Using (34) we find

$$\frac{N}{\mathscr{N}} \frac{d}{dN}(\mathscr{N}) = -(d-1) + N - \frac{(d-1)}{\mathscr{N}} \tag{36}$$

but again from (34),

$$\log \frac{1}{\mathscr{N}} = \log\left\{\frac{N^{d-1}}{(d-1)!}\right\} - N - \log\left\{1 - e^{-N} \sum_{k=0}^{d-2} \frac{N^k}{K!}\right\}. \tag{37}$$

Combining (36) and (37) and discarding terms that go to zero as $N \to \infty$ yields the following asymptotic result for the expression (35):

$$\{\psi, \phi\}[\mathscr{A}^{\text{op}}]_{\text{asymp}} = \log\left\{\frac{N^{d-1}}{(d-1)!}\right\} - (d-1) \tag{38}$$

corresponding to equality in (27). So this establishes the isotropic measurement scheme as the optimal apparatus in the large N limit, as claimed.

Note that the uniqueness, or otherwise, of this solution is not of interest to us. We simply wish to demonstrate that the fundamental limit can be realized asymptotically by some scheme. However, we do further conjecture, upon the basis of detailed calculations performed in the case $d = 2$ [1], that the isotropic scheme is uniquely the best and that its performance can be approached arbitrarily closely by finite m measurement schemes for large n.

7. Inferential uncertainty principle

Aside from solving the asymptotic OSDP, for its own sake, what we have shown is that under the best possible conditions for quantum observations, the intrinsic quantum noise embodied in the rule:

$$p(\phi^j|\psi) = |\langle\phi^j|\psi\rangle|^2 \tag{39}$$

limits confidence in the quantum state by the measure given in (22). Furthermore, it is under similarly idealized conditions possible to realize these constraints arbitrarily closely for large ensembles.

This represents an alternative, inferential uncertainty principle (IUP), since it limits the degree to which we can in principle know the initial conditions of a quantum ensemble containing N identical d state elements (where it is understood that once such information has been obtained the original ensemble has been irrevocably disturbed).

Note that the usual Heisenberg uncertainty principle (HUP) says absolutely nothing about the degree to which we may know a state. The simple way to see this is to realize that any perfectly known state satisfies this rule. It is really a constraint upon the degree to which it makes sense to speak of observables as having simultaneous values through study of constraints upon their operator dispersions. The reality of these constraints is well known from experiment.

In no sense does the IUP challenge the HUP, the two simply say different things. They are complementary and it is interesting to note that the derivation of both is rooted in the geometry of Hilbert space, through properties of the Hilbert space inner product. That of the IUP follows from the functional form of (39) when used as a building block for generating inferred distributions; while that of the HUP follows directly from the Schwartz inequality.

A future paper will deal with the relation between the usual quantum mechanical entropy and the quantum correlation information.

Acknowledgments

The bulk of this work formed part of a PhD thesis, submitted to the University of Bristol in December 1989. I thank John Hannay at Bristol for initiating my work in this area and for many useful discussions during its development. Some additional work and the writing of this paper was done at the University of Melbourne. I thank the people there, especially Professor McKellar, for that opportunity.

Appendix 1. Decomposition formula

In this appendix we prove the decomposition formula (12). To begin we shall need the simple result that for l_1, l_2, l_3 indices to some orthonormal basis $\{|l\rangle\}_{l=1}^{d}$:

$$\int \langle l_1|\omega\rangle\langle\omega|l_2\rangle f(|\langle l_3|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\bar{\omega}} = 0 \tag{40}$$

unless $l_1 = l_2$.

To show this, write $|\omega\rangle$ in polar coordinates as $\langle l|\omega\rangle = u_l\,\mathrm{e}^{\mathrm{i}\theta_l}$. Then the ray measure becomes

$$\mathrm{d}\hat{\Omega}_{\bar{\omega}} \equiv \frac{(d-1)!}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\,\mathrm{e}^{\mathrm{i}k}\prod_{l=1}^{d}\left(\int_0^{2\pi}\frac{\mathrm{d}\theta_l}{2\pi}\int_0^{\infty}\mathrm{d}u_l\,\mathrm{e}^{-\mathrm{i}ku_l}\right)$$

and the integrand is

$$\langle l_1|\omega\rangle\langle\omega|l_2\rangle f(|\langle l_3|\omega\rangle|^2) = \mathrm{e}^{\mathrm{i}(\theta_{l_1}-\theta_{l_2})}u_{l_1}u_{l_2}f(u_{l_3}).$$

Consider the θ_l integrations done first and observe that

$$\prod_{l=1}^{d}\left(\int_0^{2\pi}\frac{\mathrm{d}\theta_l}{2\pi}\right)\mathrm{e}^{\mathrm{i}(\theta_{l_1}-\theta_{l_2})} = \delta_{l_1 l_2}$$

from which the desired result follows irrespective of the nature of $f(u)$.

Consider now the integral:

$$I \equiv \int |\langle\phi|\omega\rangle|^2 f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\bar{\omega}}.$$

Insert an expansion basis chosen so that one element is equal to $|\psi\rangle$. Let this be $|d\rangle$. Then

$$I=\sum_{ll'}\int \langle\omega|l'\rangle\langle l'|\omega\rangle\langle\phi|l\rangle\langle l|\phi\rangle f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\hat{\omega}}$$

and application of (40) with $|d\rangle=|\psi\rangle$ yields:

$$I=\sum_{l=1}^{d-1}\int |\langle l|\phi\rangle|^2|\langle l|\omega\rangle|^2 f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\hat{\omega}} + \int |\langle\psi|\phi\rangle|^2|\langle\psi|\omega\rangle|^2 f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\hat{\omega}}. \tag{41}$$

The integral in the second term is already in the class to which (7) applies. Those in the first term can be made so, since with $|\psi\rangle=|d\rangle$, we have $\langle l|\psi\rangle=0$ for $l<d$. Then the symmetry of the integration measure ensures that:

$$\int |\langle l|\omega\rangle|^2 f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\hat{\omega}} = \frac{1}{d-1}\sum_{l=1}^{d-1}\int |\langle l|\omega\rangle|^2 f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\hat{\omega}} = \frac{1}{d-1}\int (1-|\langle\psi|\omega\rangle|^2) f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\hat{\omega}}.$$

Thus (41) becomes

$$I=\left(\sum_{l=1}^{d-1}|\langle l|\phi\rangle|^2\right)\int (1-|\langle\psi|\omega\rangle|^2) f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\hat{\omega}} + |\langle\psi|\phi\rangle|^2\int |\langle\psi|\omega\rangle|^2 f(|\langle\psi|\omega\rangle|^2)\,\mathrm{d}\hat{\Omega}_{\hat{\omega}}. \tag{42}$$

Then use of the $|\phi\rangle$ normalization,

$$\left(\sum_{l=1}^{d-1}|\langle l|\phi\rangle|^2\right)=1-|\langle\psi|\phi\rangle|^2$$

converts (42) to the result (12) with $g_\perp$, $g_\parallel$ and $C_\perp$, $C_\parallel$ as in (13, 14).

Appendix 2. Isotropic apparatus correlation

In this section we show why (28) is a good approximation to the large N behaviour of the idealized isotropic measurement scheme.

Label element j of basis k by ϕ_k^j. Let P_k^j be the proportion of the n trials on basis k in which this result occurs, where $\Sigma_{j=1}^d P_k^j=1$. Then $p(\Phi_N|\psi)$ can be made explicit, for arbitrary repeated measurement sets, as the exact expression:

$$p(\Phi_N|\psi)=\left(\prod_{k=1}^{m}\frac{n!}{P_k^1!\ldots P_k^d!}\right)\exp\left\{n\sum_{k=1}^{m}\sum_{j=1}^{d}P_k^j\log|\langle\phi_k^j|\psi\rangle|^2\right\}. \tag{43}$$

Observe that this is the product of m multinomial distributions in d variables, one for each basis. A standard result yields for the expectation value of P_k^j:

$$\langle P_k^j\rangle=|\langle\phi_k^j|\psi\rangle|^2$$

where ψ is the true state. We are interested in finite but large n behaviour so we must include some values of P_k^j that are off this shell. To do so we introduce the condition that:

$$P_k^j = |\langle \phi_k^j | \phi \rangle|^2 \tag{44}$$

for some ϕ, which is now considered as a generator of the allowed sets of P_k^j values. Invoking the uniformity of ϕ_k^j for the isotropic set allows us to conclude that the weight of ϕ should be chosen uniform so as to obtain the correct corresponding weight for the P_k^j.

Substituting (44) into (43) and dropping the, now uniform, combinatorial prefactor, leads to the approximate expression:

$$p(\Phi_N|\psi) \propto \exp\left\{ n \sum_{k=1}^{m} \sum_{j=1}^{d} |\langle \phi_k^j|\phi\rangle|^2 \log|\langle \phi_k^j|\psi\rangle|^2 \right\}. \tag{45}$$

Invoking again the uniformity of ϕ_k^j and taking m large then allows us to write

$$n \sum_{k=1}^{m} \sum_{j=1}^{d} |\langle \phi_k^j|\phi\rangle|^2 \log|\langle \phi_k^j|\psi\rangle|^2 \sim Nd \int |\langle \phi|\omega\rangle|^2 \log|\langle \psi|\omega\rangle|^2 \, d\hat{\Omega}_{\hat{\omega}} \tag{46}$$

from which (28) is found upon substitution of this into (45). Note that the validity of the above replacement improves with increasing N as does the ability to realize the condition that the ϕ_k^j should be uniform. This establishes (28) as a legitimate asymptotic correlation.

References

[1] Jones K R W 1989 *PhD thesis* University of Bristol

[2] Shannon C E and Weaver W 1949 *The Mathematical Theory of Communication* (Urbana: Illinois Press)

[3] Jones K R W 1990 *Preprint* University of Melbourne UM-P-90/31 OZ-90/11

[4] Wootters W K 1980 *PhD thesis* Center for Theoretical Physics Unversity of Texas at Austin

[5] Everett H III 1957 *PhD thesis* in *The Many-Worlds Interpretation of Quantum Mechanics* B De Witt and N Graham (eds) 1973 (Princeton: Princeton University Press)

[6] Jeffreys Sir H and Lady Jeffreys 1956 *Methods of Mathematical Physics* (Cambridge: Cambridge University Press) p 228